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LEAST-NORM LINEAR PROGRAMMING SOLUTION AS AN UNCONSTRAINED MINIMIZATION PROBLEM

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LEAST-NORM LINEAR PROGRAMMING SOLUTION AS AN UNCONSTRAINED MINIMIZATION PROBLEM

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ABSTRACT

It is shown that the dual of the problem of minimizing the 2-norm of the primal and dual optimal variables and slacks of a linear program, can be transformed into an unconstrained minimization of a convex, parameter-free, globally differentiable, piecewise quadratic function with a Lipschitz continuous gradient. If the slacks are not included in the norm minimization, one obtains a minimization problem with a convex, parameter-free, quadratic objective function subject to nonnegativity constraints only.

AMS (MOS) Subject Classifications: 90C05, 90C20, 90C25

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SIGNIFICANCE AND EXPLANATION

The linear programming problem is that of maximizing a linear objective function subject to linear inequalities and equalities. Such problems are usually solved by methods which move from a vertex to a higher neighboring vertex in the feasible region and terminate in a finite number of steps. In this report we show how the smallest solution of a linear program can be obtained by the completely unconstrained minimization of a valley-like smooth function. This reformulation should enable us to use other techniques to solve this fundamental problem.

LEAST-NORM LINEAR PROGRAMMING SOLUTION AS AN UNCONSTRAINED MINIMIZATION PROBLEM

O. L. Mangasarian

1. Introduction

In general the primal-dual solution to a linear program is not unique and sometimes the set of such solutions is unbounded in which case the problem is unstable [18]. It seems reasonable then that given a linear program one would be interested in finding a unique solution with some least norm property. In this work we will show that if one chooses that optimal solution which minimizes the 2-norm of the primal and dual optimal variables and slacks one is led to an unconstrained minimization of a convex, parameter-free, globally differentiable, piecewise-quadratic function with a Lipschitz continuous gradient. If the slacks are not included in the norm minimization, one obtains a minimization problem with a convex, parameter-free, quadratic objective function subject to nonnegativity constraints only. These reformulations of the original linear program can be solved by techniques other than the simplex method and will lead to a unique least-norm solution of the problem.

We shall consider the canonical linear program

$$\underset{(x,y)\ R^{n+m}}{\text{maximize}}\ c^Tx \quad \text{subject to} \quad y = -Ax + b, \quad (x,y) \geqq 0 \tag{1}$$

where c and b are given vectors in the real dimensional Euclidean spaces R^n and R^m respectively, A is a given $m \times n$ real matrix and the superscript T denotes the transpose. The dual linear program [3] associated with (1) is

$$\underset{(u,v)\ R^{m+n}}{\text{minimize}}\ b^Tu \quad \text{subject to} \quad v = A^Tu - c, \quad (u,v) \geqq 0\ . \tag{2}$$

It is well known [3] that solving <u>either</u> (1) or (2) is equivalent to solving <u>both</u> (1) and (2) which in turn is equivalent to the following linear complementarity problem [2]:

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Find (z,w) in R^{2k} such that

$$w = Mz + q \geq 0, \quad z \geq 0, \quad z^T w = q^T z = 0 \tag{3}$$

where

$$k = n + m, \quad M = \begin{pmatrix} 0 & A^T \\ -A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -c \\ b \end{pmatrix}, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad w = \begin{pmatrix} v \\ y \end{pmatrix} . \tag{4}$$

Note that M is a skew symmetric matrix satisfying $M + M^T = 0$ and hence $z^T Mz = 0$ for any z in R^k. A principal purpose of this work is to show (Theorem 1) that the least 2-norm solution $(\bar{z},\bar{w})$ of problem (3) can be obtained by solving the following unconstrained minimization problem in R^{k+1}

$$\underset{(r,\alpha)\in R^{k+1}}{\text{minimize}} \quad f(r,\alpha) \tag{5}$$

where

$$f(r,\alpha) := q^T r + \frac{1}{2} \|(M^T r - \alpha q)_+\|_2^2 + \frac{1}{2} \|(-r)_+\|_2^2 . \tag{6}$$

Here $\|\cdot\|_2$ denotes the 2-norm and for t in R^k, $(t)_+$ denotes a vector in R^k with components $((t)_+)_i = \max\{t_i, 0\}$, $i = 1,\ldots,k$, where t_i is the ith component of t. It is easy to verify that $f(r,\alpha)$ is a convex, globally differentiable function on R^{k+1}. We will also show that f has a Lipschitz continuous gradient on R^{k+1}. In fact, it will be shown that the unconstrained minimization problem (5) is equivalent to the dual of the least-norm problem

$$\underset{(z,w)\in R^{2k}}{\text{minimize}} \quad \{\frac{1}{2} \|z,w\|_2^2 \mid w = Mz + q \geq 0, \quad z \geq 0, \quad q^T z = 0\} . \tag{7}$$

We will also show the value $\bar{\alpha}$, where $(\bar{r},\bar{\alpha})$ is a solution of the unconstrained problem (5), plays an important role in interpreting the stability of least-norm solution of (3) and hence of (1)-(2) (Corollary 3).

By considering the dual of a slightly different least-norm problem

$$\underset{z\in R^k}{\text{minimize}} \quad \{\frac{1}{2} \|z\|_2^2 \mid Mz + q \geq 0, \quad z \geq 0, \quad q^T z = 0\} \tag{8}$$

we are led (Theorem 2) to the following convex quadratic minimization problem with nonnegativity constraints only

$$\begin{array}{ll} \underset{\substack{(s,t,\beta)\in R^{2k+1} \\ (s,t,\beta)\geq 0}}{\text{minimize}} & g(s,t,\beta) \end{array} \tag{9}$$

where

$$g(s,t,\beta) := q^T s + \frac{1}{2}\|M^T s - \beta q + t\|_2^2 . \tag{10}$$

Here again the value $\bar{\beta}$ of β for which (9) has a solution can be interpreted as a stability measure for the dual linear programs (1)-(2) (Corollary 5).

We note that since the conditions $Mz + q \geq 0$, $z \geq 0$ imply that $q^T z \geq 0$ it follows that in both problems (7) and (8) the last constraint $q^T z = 0$ can be written in the equivalent form $q^T z \leq 0$. Thus the Lagrange multiplier associated with $q^T z = 0$ is implicitly nonnegative.

The formulations (5) and (9) have computational implications. Thus the unconstrained minimization problem (5) can be solved by gradient and possibly other methods [8] while the quadratic problem (9) can be solved by, among others, iterative successive overrelaxation methods [11]. These aspects are discussed in Section 4.

2. The least-norm linear programming solution as an unconstrained minimization problem

We begin by establishing the following useful preliminary result.

Lemma 1. Let $S \subset R^k$, let $h:S \to R^\ell$, let $\theta:R^k \to R$ and let φ be a nondecreasing function from the nonnegative real line into R. Let $\|\cdot\|$ be any monotonic norm on R^ℓ, that is for a and b in R^ℓ, $\|a\| \leqq \|b\|$ whenever $|a_i| \leqq |b_i|$, $i = 1,\ldots,\ell$. Then

(i) $\bar{x}$ solves $\min\limits_{x\in S} \theta(x) + \varphi(\|h(x)_+\|)$ $\Longrightarrow$ $\left\langle \begin{array}{l} (\bar{x},\ \bar{y} = h(\bar{x})_+ - h(\bar{x}))\ \text{solves} \\ \min\limits_{\substack{x\in S \\ y\geqq 0}} \theta(x) + \varphi(\|h(x) + y\|) \\ \text{and the two minima are equal} \end{array} \right.$

(ii) $\left\rangle \begin{array}{l} \bar{x}\ \text{solves}\ \min\limits_{x\in S} \theta(x) + \varphi(\|h(x)_+\|) \\ \text{and the two minima are equal} \end{array} \right\rangle$ $\Longleftarrow$ $\left\langle \begin{array}{l} (\bar{x},\bar{y})\ \text{solves} \\ \min\limits_{\substack{x\in S \\ y\geqq 0}} \theta(x) + \varphi(\|h(x) + y\|) \end{array} \right.$

(iii) $\left\rangle \begin{array}{l} (\bar{x},\bar{y} = h(\bar{x})_+ - h(\bar{x}))\ \text{solves} \\ \min\limits_{\substack{x\in S \\ y\geqq 0}} \theta(x) + \varphi(\|h(x) + y\|) \end{array} \right\rangle$ $\Longleftarrow$ $\left\langle \begin{array}{l} (\bar{x},\bar{y})\ \text{solves} \\ \min\limits_{\substack{x\in S \\ y\geqq 0}} \theta(x) + \varphi(\|h(x) + y\|)\ . \end{array} \right.$

Proof. (i) Let $\bar{x}$ solve: $\min\limits_{x\in S} \theta(x) + \varphi(\|h(x)_+\|)$, let $\bar{y} = h(\bar{x})_+ - h(\bar{x})$, let $x \in S$ and let $y \geqq 0$. Then

$$\theta(x) + \varphi(\|h(x) + y\|) - \theta(\bar{x}) - \varphi(\|h(\bar{x}) + \bar{y}\|)$$

$$= \theta(x) + \varphi(\|h(x) + y\|) - \theta(\bar{x}) - \varphi(\|h(\bar{x})_+\|) \quad \text{(By definition of } \bar{y})$$

$$\geqq \theta(x) + \varphi(\|h(x) + y\|) - \theta(x) - \varphi(\|h(x)_+\|) \quad \text{(By definition of } \bar{x})$$

$$\geqq 0 \quad \text{(By norm monotonicity, nondecreasing property of } \varphi \text{ and}$$

$$|h_i(x) + y_i| \geqq h_i(x)_+, \qquad i = 1,\ldots,\ell)\ .$$

The two minima are equal by the definition of $\bar{y}$.

(ii) Let $(\bar{x},\bar{y})$ solve $\min_{x\in S, y\geqq 0} \theta(x) + \varphi(\|h(x) + y\|)$ and let $x \in S$. Then

$$\theta(\bar{x}) + \varphi(\|h(\bar{x})_+\|) \leqq \theta(\bar{x}) + \varphi(\|h(\bar{x}) + \bar{y}\|)$$

(By norm monotonicity, nondecreasing property of φ

and $|h_i(\bar{x}) + \bar{y}_i| \geqq h_i(\bar{x})_+$, $i = 1,\ldots,\ell$)

$$\leqq \theta(x) + \varphi(\|h(x) + y\|) \quad \text{for} \quad y \geqq 0$$

(By definition of $(\bar{x},\bar{y})$)

$$= \theta(x) + \varphi(\|h(x)_+\|) \quad (\text{Set } y = h(x)_+ - h(x) \geqq 0) .$$

The two minima are equal because by part (i), $\bar{x}$ and $\hat{y} = h(\bar{x})_+ - h(\bar{x})$ solve $\min_{\substack{x\in S\\ y\geqq 0}} \theta(x) + \varphi(\|h(x) + y\|)$.

(iii) This part follows by combining parts (i) and (ii) of the lemma. □

We are ready now to establish our first principal result.

<u>Theorem 1</u>. The dual linear programs (1) and (2) are solvable if and only if the unconstrained minimization problem (5) is solvable. For any solution $(\bar{r},\bar{\alpha})$ of (5) the point $(\bar{z},\bar{w})$ defined by

$$\begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} = \bar{z} := (M^T\bar{r} - \bar{\alpha}q)_+, \quad \bar{w} := (-\bar{r})_+ \tag{11}$$

is the unique point in R^{2k} which solves (3), the dual linear programs (1)-(2) and the minimum-norm problem (7). Furthermore

$$\min_{(r,\alpha)\in R^{k+1}} f(r,\alpha) = - \min_{(z,w)\in R^{2k}} \{\tfrac{1}{2}\|z,w\|_2^2 \mid w = Mz + q \geqq 0,\ z \geqq 0,\ q^Tz = 0\} . \tag{12}$$

<u>Proof</u>. If the solution set of (3) is nonempty then by quadratic programming duality [4,10] we have that

$$= \min_{(z,w)\in R^{2k}} \{\tfrac{1}{2}\|z,w\|_2^2 \,|\, w - Mz - q = 0,\ w \geqq 0,\ z \geqq 0,\ q^Tz = 0\}$$

$$= - \max_{(r,a,d,\alpha)\in R^{3k+1}} \{-\tfrac{1}{2}\|M^Tr - \alpha q + d\|_2^2 - \tfrac{1}{2}\|-r + a\|_2^2 - q^Tr \,|\, (a,d) \geqq 0\} \quad (13)$$

$$= \min_{(r,\alpha)\in R^{k+1}} \tfrac{1}{2}\|(M^Tr - \alpha q)_+\|_2^2 + \tfrac{1}{2}\|(-r)_+\|_2^2 + q^Tr \quad (5)$$

(By Lemma 1)

$$= \min_{(r,\alpha)\in R^{k+1}} f(r,\alpha) .$$

Suppose first that the dual linear programs (1)-(2) are solvable. Then the solution set of (3) is nonempty and contains a unique $(\bar{z},\bar{w})$ in R^{2k} that solves (7). By Dorn's duality theorem [4,10 Theorem 8.2.4] there exists $(\bar{r},\bar{a},\bar{d},\bar{\alpha})$ in R^{3k+1} that solves (13) above and by Lemma 1, $(\bar{r},\bar{\alpha})$ solves (5) and

$$-\tfrac{1}{2}\|\bar{z},\bar{w}\|_2^2 = f(\bar{r},\bar{\alpha}) .$$

Conversely now suppose that $(\bar{r},\bar{\alpha})$ solves (5). Then again by Lemma 1, $(\bar{r},\bar{\alpha})$ and

$$\bar{d} = (M^T\bar{r} - \bar{\alpha}q)_+ - (M^T\bar{r} - \bar{\alpha}q), \quad \bar{a} = (-\bar{r})_+ - (-\bar{r})$$

solve (13) and by Dorn's strict converse duality theorem [4,10 Theorem 8.2.5] .

$$\bar{z} = M^T\bar{r} - \bar{\alpha}q + \bar{d} = (M^T\bar{r} - \bar{\alpha}q)_+$$
$$\bar{w} = -\bar{r} + \bar{a} = (-\bar{r})_+$$

is the unique point in R^{2k} which solves (7) and hence the dual linear programs (1)-(2). We again have that

$$f(\bar{r},\bar{\alpha}) = -\tfrac{1}{2}\|\bar{z},\bar{w}\|_2^2 . \qquad \square$$

Because problem (5) is an unconstrained minimization problem with a convex differentiable objective function its solution can be achieved by setting the gradient $\nabla f(r,\alpha)$ equal to zero. We thus have the following.

Corollary 1. The dual linear programs (1) and (2) are solvable if and only if there exist $(\bar{r},\bar{\alpha}) \in R^{k+1}$ satisfying

$$\nabla f(r,\alpha) = \begin{cases} q + M(M^T\bar{r} - \bar{\alpha}q)_+ - (-\bar{r})_+ = 0 & (14) \\ -q^T(M^T\bar{r} - \bar{\alpha}q)_+ = 0 . & (15) \end{cases}$$

For any $(\bar{r},\bar{\alpha})$ satisfying (14)-(15) the point $(\bar{z},\bar{w})$ defined by (11) is the unique point in R^{2k} which solves (3), the dual linear programs (1)-(2) and the minimum-norm problem (7).

An interesting property of the unconstrained minimization problem (5) is that the minimization over the variable α can be dispensed with if α is chosen sufficiently large. This can be established by using the perturbation results of linear programming [13] as follows.

Corollary 2. Let $\{z \mid Mz + q \geqq 0,\ z \geqq 0\}$ be nonempty and let f be defined by (6). There exists an $\tilde{\alpha} \geqq 0$ such that

$$\min_{(r,\alpha)\in R^{k+1}} f(r,\alpha) = \varphi(\alpha) \quad \text{for } \alpha \geqq \tilde{\alpha} \tag{16}$$

where

$$\varphi(\alpha) := \min_{r\in R^k} f(r,\alpha) \tag{17}$$

is well defined, continuous and convex for all real α and

$$\varphi(0) = -\min_{(z,w)\in R^{2k}} \{\tfrac{1}{2}\|z,w\|_2^2 \mid w = Mz + q \geqq 0,\quad z \geqq 0\}\ , \tag{18}$$

$$\varphi(\alpha) = -\min_{(z,w)\in R^{2k}} \{\tfrac{1}{2}\|z,w\|_2^2 \mid w = Mz + q \geqq 0,\ z \geqq 0,\ q^Tz = 0\} \quad \text{for } \alpha \geqq \tilde{\alpha}\ . \tag{19}$$

Proof.

$$\min_{(r,\alpha)\in R^{k+1}} f(r,\alpha) = -\min_{(z,w)\in R^{2k}} \{\frac{1}{2}\|z,w\|_2^2 \mid w = Mz + q \geqq 0,\ z \geqq 0,\ q^Tz \leqq 0\} \qquad (20)$$

(By (12))

$$= -\min_{(z,w)\in R^{2k}} \{\alpha q^Tz + \frac{1}{2}\|z,w\|_2^2 \mid w = Mz + q \geqq 0,\ z \geqq 0\} \qquad (21)$$

for $\alpha \geqq \tilde{\alpha}$ for some $\tilde{\alpha} \geqq 0$ (By [13, Theorem 1] and since $\min_{z,w}\{q^Tz \mid w = Mz + q \geqq 0,\ z \geqq 0\} = 0$)

$$= -\max_{(r,a,d)\in R^{3k}} \{-\frac{1}{2}\|M^Tr-\alpha q+d\|_2^2 - \frac{1}{2}\|-r+a\|_2^2 - q^Tr \mid (a,d) \geqq 0\} \qquad (22)$$

(By quadratic programming duality [4,10])

$$= \min_{r\in R^k} f(r,\alpha) \qquad \text{(By Lemma 1)} \qquad (23)$$

$$= \varphi(\alpha)\ . \qquad (24)$$

This establishes (16) and (19). Note that in the above equalities the restriction of α to $\alpha \geqq \tilde{\alpha}$ is only needed for the equality between (20) and (21), otherwise for the equalities between (21) and up to (24) α is unrestricted. Because the quadratic program (21) is feasible and its dual (22) is also feasible it follows that the objective function of (21) is bounded below [4,10 Theorem 8.2.3] and hence (21) has a solution for each real α [5]. By virtue of the equality between (21) and (24) for all real α it follows that $\varphi(\alpha)$ is well defined for all real α. That $\varphi(\alpha)$ is convex and continuous for all real α follows from the parametric problem representation (21) of $\varphi(\alpha)$ [9, Theorem 1]. Finally (18) follows by setting $\alpha = 0$ in (21). $\square$

By using results of perturbation theory of convex programs [6,19] we can give a useful stability interpretation of $\bar{\alpha}$, where $(\bar{r},\bar{\alpha})$ solves (5).

Corollary 3. Let the linear program (1) be solvable. For $\delta \geqq 0$ let $(z(\delta),w(\delta))$ be the unique solution of the following perturbation of the minimum-norm problem (7)

$$\min_{(z,w)\in R^{2k}} \{\tfrac{1}{2}\|z,w\|_2^2 \mid w = Mz + q \geqq 0, \quad z \geqq 0, \quad q^T z \leqq \delta\} . \tag{25}$$

Then for any $\bar{\alpha}$ such that $(\bar{r},\bar{\alpha})$ solves (5)

$$0 \leqq \|z(0),w(0)\|_2^2 - \|z(\delta),w(\delta)\|_2^2 \leqq 2\bar{\alpha}\delta \tag{26}$$

and if $q \neq 0$

$$0 \leqq \|z(0),w(0)\|_2 - \|z(\delta),w(\delta)\|_2 \leqq \frac{\bar{\alpha}\delta}{\|z(0),w(0)\|_2} . \tag{27}$$

<u>Proof</u>. The first inequality of (26) is obvious because the feasible region of (25) for $\delta \geqq 0$ contains the feasible region for $\delta = 0$. The second inequality of (26) follows from the standard result of perturbation theory for convex programs [6 Theorem 1, 19 Theorem 29.1] that $-\bar{\alpha}$, which is the negative of an optimal multiplier of (25) associated with $q^T z \leqq \delta$ for $\delta = 0$, is a subgradient of the convex function $\frac{1}{2}\|z(\delta),w(\delta)\|_2^2$ at $\delta = 0$.

To establish (27) we note that for $\delta \geqq 0$, $(z(\delta),w(\delta))$ is also the unique solution of the convex problem

$$\min_{(z,w)\in R^{2k}} \{\|z,w\|_2 \mid w = Mz + q \geqq 0, \quad z \geqq 0, \quad q^T z \leqq \delta\} \tag{28}$$

and if $q \neq 0$, then $\bar{\alpha}/\|z(0),w(0)\|_2$ is an optimal multiplier associated with $q^T z \leqq \delta$ for $\delta = 0$ in the optimization problem (28). □

Inequality (26) states that for an error of no more than δ in satisfying the equality condition between the primal and dual objective functions, that is $0 \leqq -c^T x + b^T u = q^T z \leqq \delta$, the square of the 2-norm of the smallest optimal vector of primal and dual variables and slacks of the linear program (1) differs by no more than $2\bar{\alpha}\delta$ from the square of the 2-norm of the corresponding optimal vector for which $q^T z = 0$. It follows that the smaller $\bar{\alpha}$ is the more stable is the linear program (1) under errors in its minimum value. Linear programs with large $\bar{\alpha}$ would in general be harder to solve than those with small $\bar{\alpha}$. Computational experience in [12] where a perturbation parameter ϵ was used which is related to $\frac{1}{\alpha}$ bears out this observation. The least value $\hat{\alpha}$ such that $(\hat{r},\hat{\alpha})$ is a solution of (5) may be thought of as a unique <u>stability parameter</u> associated with the linear program (1), and because of (16), it may be defined as $\min\{\alpha | \varphi(\alpha) = \varphi(\tilde{\alpha}),\ 0 \leqq \alpha \leqq \tilde{\alpha}\}$. Because the convex set $\{\alpha | \varphi(\alpha) = \varphi(\tilde{\alpha}),\ 0 \leqq \alpha \leqq \tilde{\alpha}\}$ is compact, in fact a closed line interval, $\hat{\alpha}$ is well defined. By the equivalence between (20) and (24), $\hat{\alpha}$ can also be defined as the least nonnegative multiplier associated with the constraint $q^T z \leqq 0$ in the quadratic program (20). Hence (26) and (27) can be sharpened by replacing $\bar{\alpha}$ by $\hat{\alpha}$. Related but different results for perturbations of linear programs are given in [17].

Going back to the unconstrained minimization problem (5), it is straightforward to show that its objective function $f(r,\alpha)$ has a Lipschitz continuous gradient if we make use of the following property of monotonic norms due to Y.-C. Cheng [1].

<u>Lemma 2</u> (Cheng). For any monotonic norm $\|\cdot\|$ on R^ℓ and any a and b in R^ℓ it follows that

$$\|a_+ - b_+\| \leqq \|a - b\| .$$

This lemma is a direct consequence of the inequality

$$|a_i - b_i| - |(a_i)_+ - (b_i)_+| \geqq 0, \quad i = 1,\ldots,\ell$$

and the norm monotonicity. The Lipschitz continuity of $\nabla f(r,\alpha)$ can be easily established by using Lemma 2 and the dual vector norm $\|\cdot\|'$ associated with any given vector norm $\|\cdot\|$ on R^ℓ and defined by $\|x\|' = \sup\limits_{\substack{y \in R^\ell \\ \|y\|=1}} x^T y$ for x in R^ℓ and from which follows the generalized Cauchy inequality $x^T y \leqq \|x\| \cdot \|y\|'$ for all x and y in R^ℓ. For $\infty \geqq p,\ q \geqq 1$ and $(1/p) + (1/q) = 1$ and any x in R^ℓ, the p-norm $\|x\|_p := (\sum_{i=1}^{\ell} |x_i|^p)^{1/p}$ and the q-norm $\|x\|_q$ are monotonic and dual norms to each other [7,19]. By using the generalized Cauchy inequality and Lemma 2 it is a straightforward algebraic exercise to obtain the following.

<u>Lemma 3</u>. Let $\|\cdot\|$ be any monotonic norm on R^{k+1} and let $\|\cdot\|'$ be its dual norm. Then

$$\|\nabla f(r^1,\alpha^1) - \nabla f(r^2,\alpha^2)\| \leqq L\|(r^1,\alpha^1) - (r^2,\alpha^2)\|$$

for all (r^1,α^1) and (r^2,α^2) in R^{k+1} where

$$L = 1 + \|M\|^2 + \|M\|(\|q\| + \|q\|') + \|q\| \cdot \|q\|' \ . \tag{29}$$

For the 2-norm $\|\cdot\|_2$

$$L = L_2 = 1 + (\|M\|_2 + \|q\|_2)^2 \ . \tag{30}$$

3. The least-norm linear programming solution as the minimization of a nonnegatively constrained quadratic function

An entirely analogous development to that of Section 2 but which is based on (8) rather than (7) leads to the following results the proofs of which are omitted.

Theorem 2. The dual linear programs (1) and (2) are solvable if and only if the quadratic program (9) is solvable. For any solution $(\bar{s},\bar{t},\bar{\beta})$ of (10) the point $(\bar{z},\bar{w})$ defined by

$$\begin{pmatrix}\bar{x}\\ \bar{u}\end{pmatrix} = \bar{z} := M^T\bar{s} - \bar{\beta}q + \bar{t}, \quad \bar{w} := M\bar{z} + q \tag{31}$$

is the unique point in R^{2k} which solves (3), the dual linear programs (1)-(2) and the minimum-norm problem (8). Furthermore

$$\min_{\substack{(s,t,\beta)\in R^{2k+1}\\ (s,t,\beta)\geqq 0}} g(s,t,\beta) = -\min_{(z,w)\in R^{2k}} \{\tfrac{1}{2}\|z\|_2^2 \mid w = Mz + q,\ z \geqq 0,\ q^Tz = 0\} \ . \tag{32}$$

Corollary 4. Let $\{z \mid Mz + q \geqq 0,\ z \geqq 0\}$ be nonempty and let g be defined by (10). There exists a $\tilde{\beta} \geqq 0$ such that

$$\min_{\substack{(s,t,\beta)\in R^{2k+1}\\ (s,t,\beta)\geqq 0}} g(s,t,\beta) = \psi(\beta) \quad \text{for} \quad \beta \geqq \tilde{\beta} \tag{33}$$

where

$$\psi(\beta) := \min_{\substack{(s,t)\in R^{2k}\\ (s,t)\geqq 0}} g(s,t,\beta)$$

is well defined, continuous and convex for all real β and

$$\psi(0) = -\min_{z\in R^k} \{\tfrac{1}{2}\|z\|_2^2 \mid Mz + q \geqq 0,\ z \geqq 0\} \ ,$$

$$\psi(\beta) = -\min_{z\in R^k} \{\tfrac{1}{2}\|z\|_2^2 \mid Mz + q \geqq 0,\ z \geqq 0,\ q^Tz = 0\} \quad \text{for} \quad \beta \geqq \tilde{\beta} \ .$$

<u>Corollary 5</u>. Let the linear program (1) be solvable. For $\delta \geq 0$ let $z(\delta)$ be the unique solution of the following perturbation of the minimum norm problem (8)

$$\min_{z \in R^k} \{\frac{1}{2} \|z\|_2^2 | Mz + q \geq 0,\ z \geq 0,\ q^T z \leq \delta\} .$$

Then for any $\bar{\beta}$ such that $(\bar{s},\bar{t},\bar{\beta})$ solves (9)

$$0 \leq \|z(0)\|_2^2 - \|z(\delta)\|_2^2 \leq 2\bar{\beta}\delta$$

and if $q \not\geq 0$

$$0 \leq \|z(0)\|_2 - \|z(\delta)\|_2 \leq \frac{\bar{\beta}\delta}{\|z(0)\|_2} .$$

The least value $\hat{\beta}$ such that $(\hat{s},\hat{t},\hat{\beta})$ is a solution of (9) may also be considered a stability parameter for the linear program (1) and may be defined as $\min\{\beta | \psi(\beta) = \psi(\tilde{\beta}),\ 0 \leq \beta \leq \tilde{\beta}\}$ where $\tilde{\beta}$ is defined in (33). Alternatively $\hat{\beta}$ may be taken as the least nonnegative multiplier associated with the constraint $q^T z \leq 0$ in the quadratic program (8).

4. Computational implications

The reformulation of the linear program (1) as an unconstrained minimization problem (5) or as a quadratic program (9) is not merely of theoretical interest but may also have computational potential. For example, the simple gradient algorithm [8,16]

$$\left.\begin{array}{l} (r^{i+1},\alpha^{i+1}) = (r^i,\alpha^i) - \gamma\nabla f(r^i,\alpha^i), \quad i = 0,1,\ldots \\ \text{where} \\ 0 < \gamma < 2/L_2 \quad \text{and} \quad L_2 = 1 + (\|M\|_2 + \|q\|_2)^2 \end{array}\right\rangle \tag{34}$$

will generate a sequence $\{(r^i,\alpha^i)\}$, $i = 0,1,\ldots,$ in R^{k+1} starting from any point (r^0,α^0) in R^{k+1} which will converge to a solution $(\bar{r},\bar{\alpha})$ of (5) [16] if the linear program (1) has a solution. The convergence however may be slow and the error is bounded by [16]

$$\varepsilon^i \leqq \frac{1}{(\varepsilon^0)^{-1} + \mu i} \tag{35}$$

where $\varepsilon^i := f(r^i,\alpha^i) - f(\bar{r},\bar{\alpha})$ *and* $\mu = \dfrac{\gamma(2 - \gamma L_2)}{2\|(r^0,\alpha^0) - (\bar{r},\bar{\alpha})\|_2^2}$. It would be very interesting to develop faster and possibly finite methods for solving (5).

Similarly problem (9) may be solved by the successive overrelaxation (SOR) or other iterative methods of [11]. However, one would be working in the space $R^{2k+1} = R^{2(m+n)+1}$ which is of higher dimension than that of the space R^{n+m} of (1). Other SOR methods for solving linear programs have been developed elsewhere [12] which do not enlarge the space of the original problem, however they contain an unknown but finite perturbation or penalty parameter. The formulation (9) gets rid of the parameter at the expense of enlarging the space of the problem. Another possible method for solving (9) is the conjugate gradient method [15,14].

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It is shown that the dual of the problem of minimizing the 2-norm of the primal and dual optimal variables and slacks of a linear program, can be transformed into an unconstrained minimization of a convex, parameter-free, globally differentiable, piecewise quadratic function with a Lipschitz continuous gradient. If the slacks are not included in the norm minimization, one obtains a minimization problem with a convex, parameter-free, quadratic objective function subject to nonnegativity constraints only.

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